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Department of Engineering Science
Aerospace Engineering and Nuclear Engineering

FACULTY OF ENGINEERING AND APPLIED SCIENCES

State University of New York at Buffalo

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BOUNDARY INTEGRAL EQUATION METHODS APPLIED TO WAVE PROBLEMS

by

Richard Paul Shaw, Professor

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Boundary Integral Equation Methods Applied

to Wave Problems

Richard Paul Shaw, Professor
Department of Engineering Science, Aerospace Engineering
and Nuclear Engineering
Department of Civil Engineering
State University of New York at Buffalo

SUMMARY

An introduction is given to the application of boundary integral equation methods to the areas of wave propagation, radiation and scattering. Such applications were among the first for this method which made use of the power of modern digital computers; however, there is a long history of integral methods in this field. The emphasis here is on fundamental concepts coupled with illustrations of the application of this method. The breadth of the applications of this approach precludes any attempt at overall completeness and many details are left to an extensive, but undoubtedly incomplete, list of references. There is however sufficient material to formulate the solution of most problems, with some of the inevitable pitfalls indicated.

INTRODUCTION

In view of the role of integral equations in the development of wave theory, it is not strange that some of the first applications of boundary integral equation methods were to examples in wave propagation and scattering. The use of Green's functions in all branches of mathematical physics is pervasive enough that individual references are scarcely required; one readily available reference which includes a wide range of applications of Green's functions is Morse and Feshbach (1953). A review of Green's function techniques used by the mathematical physics community at that time, as exemplified by this text, would however reveal that this use was primarily either to obtain analytical (exact) solutions to a fairly restricted class of problems (e.g., those simple geometries and boundary conditions for which a particular Green's function could be constructed so as to reduce Green's Theorem to a simple guadrature) or to obtain asymptotic expansion solutions. A third form of application lay in the development of various integral identities, existence and uniqueness proofs etc.; these have played a major role in the development of general theory.

The use of boundary integral equations to solve wave problems numerically is a fairly recent development with the first papers using computer solutions based on the boundary integral equation method for transient acoustic wave scattering appearing in the early 1960's, e.g., Friedman and Shaw (1962), Shaw and Friedman (1962), followed by a similar treatment of time harmonic elastic and acoustic wave scattering, Banaugh and Goldsmith (1963). Prior to this time, there was of course a great deal of work done on these forms of equations, e.g., Kupradse (1935), but these papers appear to mark the shift to computer

solutions. At this same time, Hess (1962) was developing computer solutions to potential flow problems and Jaswon (1963), Symm (1963) for other potential problems. Since that time, the BIE method has been employed (in some cases, rediscovered) as a numerical solution method by researchers in many areas of wave research. Before going into details of individual cases, it is useful to separate these problems into sub-groups by recognizing some distinctive characteristics. One separation is that of transient versus time harmonic behavior; both are of great physical importance in many areas of physics and engineering but require quite different formulations and solution methods. Another division is that between exterior and interior problems; for the former, solutions are sought in an infinite domain exterior to some prescribed surface while the latter requires solution in a bounded region interior to such a surface. (This distinction blurs somewhat for surfaces of infinite extent). Appropriate boundary conditions are specified on the surface in either case with the exterior problem requiring some further specification of 'behavior at infinity'. A further grouping may be made on the basis of the inhomogeneity or forcing function. A problem driven by some prescribed boundary condition, e.g., a given displacement or velocity field on a specified surface would be considered a mechanical radiation problem. A problem wherein an obstacle with a prescribed boundary condition (usually homogeneous) interacts with some incident wave field generated by sources elsewhere (frequently considered as plane waves coming from 'infinity') would be called a scattering problem. Of interest as well is the unforced or eigenvalue problem. Still further distinction may be made between two and three spatial dimensional problems; this

distinction being straightforward in time harmonic problems but presenting a serious difficulty in transient problems. Other categorization may be made on the basis of the type of boundary conditions prescribed (Dirichlet, Neumann, Robin, etc.). Finally, and possibly most important physically, is the particular medium through which the waves propagate, e.g., acoustic, elastic, electromagnetic, water (especially free surface gravity waves), and thus the physical significance of the dependent variables and their boundary and initial conditions. Fortunately, the boundary integral equation approach is applicable to all of these 'variations on a theme'. Its primary limitations are the requirement of a known fundamental solution for the particular governing equation and linearity of the equation, but not necessarily of the boundary conditions. Emphasis here will be given to mechanical rather than electromagnetic wave problems; integral equation approaches to the latter are discussed at length by Mittra (1973).

The bulk of the research based on boundary integral equation methods applied to wave problems has been directed towards numerical solution, using large scale digital computers and this will form the primary theme in this chapter. Since efficient algorithms exist for the solution of systems of algebraic equations, the most common procedure is to reduce the governing boundary integral equation to such a system by approximating the boundary values of the dependent variable by suitable trial functions over selected boundary elements. The nature of the boundary integral equation approach is to reduce the number of approximating dependent variables, e.g., trial function coefficients, by only requiring boundary values of the original dependent variable as opposed

to other solution schemes which require a solution everywhere in the domain of the problem. Such a procedure is particularly useful in exterior problems where other numerical solution methods require some 'finite' closing surface at which their approximating mesh may be terminated, e.g., see Vastano and Reid (1967) and Shaw (1974) for a direct comparison of exterior solutions by finite differences and BIE respectively. The price paid for this reduction is a full coefficient matrix as opposed to the sparse matrices found in corresponding finite difference or finite element methods. Some overall reviews of this approach are available; Schenck (1967) for time harmonic acoustic scattering and radiation, Shaw (1970) for transient and time harmonic acoustic scattering and radiation, Chertock (1971) and Kleinman and Roach (1974) again for Helmholtz problems and Mittra (1973) for the electromagnetic case. In addition, the books by Kupradze (1935), (1963) discuss integral equation formulations for elastodynamics, while Pao and Mow (1973) give a section on numerical solution for elastic scattering problems using boundary integral equations. It is clear, however, the boundary integral equation method is <u>more</u> than a numerical technique; it is a reformulation completely equivalent to any original partial differential equation representation. As such, it lends itself to other forms of solution, i.e., analytical and approximate as well as numerical. Analytical solutions are widely covered in classical mathematical physics, e.g., Morse and Feshbach (1953), and will not be pursued further here. The distinction between approximate and numerical solutions is one of semantics since clearly numerical solutions are also approximate in some sense. What is meant here is that approximate solutions will be taken as global or functional solutions to a system

of equations which represent some form of expansion of the original equation in terms of a physical parameter which characterizes the problem. Numerical solutions on the other hand will be solutions to the original equation with some piecewise and local approximate representation of the dependent variable. Thus in the former, it is the governing equation which is 'approximated' while in the latter it is the dependent variable. Typical of such approximate techniques are the high frequency asymptotic expansions of classical optics, e.g., Northover (1971). Also of interest are expansions in terms of geometrical parameters, for example the offset distance in shells of slightly nonconcentric circular or other simple boundaries, Shaw and Tai (1974), Shaw (1978a) or the plate thickness in the theory of thin elastic plates, Shaw (1978b). Examples of the latter approach will be discussed in a later section. Solution schemes based on Galerkin or related moment methods can be employed as well in the solution of integral equations, e.g., Delves and Walsh (1974), Fox (1962) for general theory and Sharma (1967) for a specific example. Indeed the piecewise numerical approximation mentioned above could be thought of as a special case of a Galerkin approach with particularly simple trial functions.

FORMULATION

Wave propagation problems arise in so many areas of mathematical physics and engineering sciences that it is important to concentrate on their similarities, which are primarily mathematical, rather than their differences, which are primarily physical. In some cases, the physical differences require a difference in formulation and these will be treated as they arise. In general however, wave propagation

problems may be classified as hyperbolic (time dependent) or elliptic (time harmonic) differential systems. The most common governing equations of these forms are the wave equation

$$\nabla^2 \psi(\bar{r},t) = c^{-2}\, \partial^2 \psi(\bar{r},t)/\partial t^2 \tag{1}$$

and the Helmholtz equation (with a time harmonic behavior required) i.e., $\psi(\bar{r},t) = \phi(\bar{r})\exp[-i\omega t]$

$$\nabla^2 \phi(\bar{r}) + k^2 \phi(\bar{r}) = 0 \tag{2}$$

respectively with c the phase speed and $k = \omega/c$ the wave number.

The integral equation formulation for the wave equation, developed in 1882 by Kirchhoff, has been widely interpreted as a mathematical formulation of an even earlier physical statement, Huygens principle, e.g., see Baker and Copson (1939). While the form of Kirchhoff's integral formulation can be derived directly by an application of Green's Theorem in four dimensional space time, e.g., Friedman and Shaw (1962), it follows most readily albeit in a more restricted sense from Helmholtz's integral formulation; this shall be the approach here.

Consider the Helmholtz equation with an inhomogeneous term, applied to a domain D with a boundary B upon which some appropriate boundary conditions on ϕ (e.g., Dirichlet, Neumann, Robin) are specified. The fundamental or point source or infinite space Green's function solution (these terms shall be used here interchangeably) satisfies (using ∇_o^2 to represent the Laplacian in $\bar{r}_o$ coordinates)

$$\nabla_o^2 G(\bar{r};\bar{r}_o) + k^2 G(\bar{r};\bar{r}_o) = -4\pi\,\delta(\bar{r}-\bar{r}_o) \tag{3}$$

and may be found as

$$G(\bar{r};\bar{r}_o) = G(|\bar{r}-\bar{r}_o|) = \exp[ikR]/R \tag{4}$$

where $R \equiv |\bar{r} - \bar{r}_o|$. The sign in the exponential term is chosen to insure outgoing waves from the source with a time behavior of the form $\exp(-i\omega t)$. The unknown ϕ satisfies

$$\nabla_o^2 \phi(\bar{r}_o) + k^2 \phi(\bar{r}_o) = -4\pi f(\bar{r}_o) \tag{5}$$

Equation (5) multiplied by G minus equation (4) multiplied by ϕ may be integrated (in $\bar{r}_o$ coordinates) over the domain of interest D and the resulting integral by Green's Theorem reduced to an integral over the surface B which forms the boundary of D . If any portion of D is infinite such as to require a boundary at infinity, appropriate limiting conditions must be included for this portion of B (this is not difficult for most wave problems but can require case in some cases, particularly in some water wave applications. This results in the inhomogeneous Helmholtz integral equation, e.g., Morse and Feshbach, sec. 7.2 (1973)

$$\varepsilon\,\phi(\bar{r}) = \int_D f(\bar{r}_o)\,G(R)\,dV_o + (1/4\pi)\int_B \left[G(R)\,\nabla_o\phi(\bar{r}_o) - \phi(\bar{r}_o)\,\nabla_o G(R)\right]\cdot d\bar{A}_o \tag{6}$$

with $d\bar{A}_o$ and dV_o a directed area element and a volume element respectively. The first integral represents the contribution of the inhomogeneous or forcing function given in the domain D ; the second represents the contribution of the boundary values with the area element $d\bar{A}_o$ positive away (outward) from the domain D . The gradient operator ∇_o is expressed in $\bar{r}_o$ coordinates which will be referred to as source or integration point coordinates; $\phi(\bar{r})$ is evaluated at $\bar{r}$ which will be called the field point. The parameter ε is zero if the field point is exterior to the domain D (i.e., the delta function $\delta(\bar{r} - \bar{r}_o)$ is zero everywhere within D) and is one if the field point lies within the domain D . Of particular intereet to boundary integral equation methods is the case where the field point lies on the boundary B. For this case, the boundary integral has a singular contribution at the point $R = 0$ which may be evaluated by distorting the actual boundary to exclude the field point from the domain of integration and taking a limiting process as the distorted boundary returns to the original form. For a plane boundary surface, this results in a value of 1/2 for ε ; for a field point at a corner, the appropriate choice for ε is $(1/4\pi)$ (solid angle of the domain excluded), e.g., Banaugh and Goldsmith (1963)-Appendix C. Furthermore, the boundary integral must be taken as aprincipal value integral, i.e., the singular point $R = 0$ is to be excluded from the integration (since its effect has already been included in the modification of ε to some value other than zero or one). Equation (6) may be written as

$$(7)\quad \varepsilon\phi(\bar{r}) = \int_D \left\{ f(\bar{r}_o) \exp[ikR]/R \right\} dV_o$$
$$+ (1/4\pi) \int_B \left\{ (1/R) \exp[ikR]\, \partial\phi(\bar{r}_o)/\partial n_o - \phi(\bar{r}_o) \exp[ikR] \cdot [ik/R - 1/R^2]\, \partial R/\partial n_o \right\} dA_o$$

where n_o is the outward normal (from D). If the time harmonic dependence is reintroduced, this equation becomes

$$\varepsilon\,\phi(\bar{r})\exp[-i\omega t] = \int_D \{ f(\bar{r}_0)\exp[-i\omega(t-R/c)]/R \}\, dV_0 \tag{8}$$
$$+ (1/4\pi)\int_B \{ (1/R)\,\partial\phi(\bar{r}_0)/\partial n_0 - \phi(\bar{r}_0)[i\omega/cR - 1/R^2]\,\partial R/\partial n_0 \}$$
$$\cdot \exp[-i\omega(t-R/c)]\, dA_0$$

These terms may be thought of as representing the inversion of a Fourier transform in time of the time dependent functions $\psi(\bar{r},t)$ and $F(\bar{r},t)$ which satisfy:

$$\nabla^2\psi(\bar{r},t) - (1/c^2)\,\partial^2\psi(\bar{r},t)/\partial t^2 = -4\pi F(\bar{r},t) \tag{9}$$

or upon transformation

$$\nabla^2\tilde{\psi}(\bar{r};\omega) + k^2\tilde{\psi}(\bar{r};\omega) = -4\pi\tilde{F}(\bar{r};\omega) \tag{10}$$

with $\tilde{\psi}(\bar{r};\omega)$ identified with ϕ and $\tilde{F}(\bar{r};\omega)$ with f, i.e.,

$$\tilde{\psi}(\bar{r};\omega) = \int_{-\infty}^{\infty}\psi(\bar{r},t)\exp[i\omega t]\,dt = \phi(\bar{r})$$

$$\psi(\bar{r},t) = (1/2\pi)\int_{-\infty}^{\infty}\phi(\bar{r})\exp[-i\omega t]\,d\omega$$

Then

$$\varepsilon\,\psi(\bar{r},t) = \int_D (1/2\pi)\int_{-\infty}^{\infty}\{ f(\bar{r}_0)/R \}\exp[-i\omega(t-R/c)]\,d\omega\, dV_0 \tag{11}$$
$$+ (1/4\pi)\int_B (1/2\pi)\int_{-\infty}^{\infty}\{ (1/R)\,\partial\phi(\bar{r}_0)/\partial n_0 - \phi(\bar{r}_0)[i\omega/cR - 1/R^2]\,\partial R/\partial n_0 \}$$
$$\cdot \exp[-i\omega(t-R/c)]\,d\omega\, dA_0$$

These integrals may be readily interpreted as time retarded values, i.e., by means of the shifting theorem in Fourier transform theory. f, ϕ and $\partial\phi/\partial n_o$ do not contain ω explicitly and the single appearance of ω within the integral may be interpreted as the transform of a time derivative. Thus, using a retarded time $t_o = t - R/c$, this inversion leads to

$$\text{(12)} \quad \epsilon\,\psi(\bar{r},t) = \int_D \{F(\bar{r}_o,t_o)/R\}\, dV_o$$

$$+ (1/4\pi) \int_B \{(1/R)\,\partial\psi(\bar{r}_o,t_o)/\partial n_o + [(1/cR)\,\partial\psi(\bar{r}_o,t_o)/\partial t_o + \psi(\bar{r}_o,t_o)/R^2]\,\partial R/\partial n_o\}\, dA_o$$

which is Kirchhoff's 'retarded time' integral equation, equivalent to the original wave equation. The concept of a retarded time arises naturally in the sense that an event occurring at a source point $\bar{r}_o$ at time t_o is felt at a field point $\bar{r}$ only at time t where $t = t_o + R/c$. As mentioned above, the restrictions on this derivation, e.g., that $\psi(\bar{r},t)$ have a Fourier transform zero initial values, limits of integration over B which do not depend on t etc., may be removed by a direct derivation, e.g., Morse and Feshbach (1953) using singular solutions or Friedman and Shaw (1962) using a four-dimensional Green's Theorem.

These two integral equations, equations (6) and (12) form the basis of the boundary integral equation method. The three-dimensional Helmholtz integral equation may be reduced to a two-dimensional form called the Weber integral equation without difficulty;

$$\text{(13)} \quad \epsilon\,\phi(\bar{r}) = (i/4) \int_B [H_o^{(1)}(kR)\,\partial\phi(\bar{r}_o)/\partial n_o - \phi(\bar{r}_o)\,\partial[H_o^{(1)}(kR)]/\partial n_o]\, dA_o + (i/4) \int_D H_o^{(1)}(kR)\, f(\bar{r}_o)\, dA_o$$

with B is now the one-dimensional boundary curve of the two dimensional domain D , and ds_o , dA_o are line element and an area element respectively.

The Kirchhoff retarded time integral equation cannot be reduced simply for two-dimensional problems. This is due to the inherent difference in the physical nature of a transient point source solution in two and in three dimensions, e.g., Hadamard, (1952). The two-dimensional point source solution requires an additional integration over time which means that two dimensional transient problems may just as well be left as three dimensional problems for cylindrical surfaces (the two-dimensional domain extended as a cylinder into the third (z) direction) with no direct z dependence of the dependent variables. Due to the appearance of R within the retarded time, t_o , there will be a variation of $\phi(\bar{r}_o,t_o)$ in the z_o direction; the integration over then plays a role equivalent to the time integration for the two-dimensional point source, e.g., Friedman and Shaw (1962).

The forms of these two formulations indicate a significant difference in their physical nature and thus in their numerical solution. The time harmonic case, i.e., the Helmholtz or Weber integral equations, leads to a Fredholm equation which upon approximation by suitable trial functions and boundary elements leads to a system of simultaneous algebraic equations (linear for linear boundary conditions). The transient case leads to a Volterra integral equation; this may not appear obvious from the form of the Kirchhoff retarded time equation, but taking account of the general initial value nature of such problems (and a corresponding null field prior to such values, i.e., for $t < 0$) there will be portions of the boundary surface sufficiently far away

from the field point so as to have negative retarded time values, i.e., are evaluated at times when the field there has zero values and thus do not contribute, at least for some interval of time. This in turn implies that the limits of integration in the integral depend on time, increasing as time increases, i.e., a Volterra integral equation. Upon suitable numerical approximation, these equations lead to successive, rather than simultaneous, algebraic equation, thereby considerably simplifying the solution procedure. For finite bodies, there will come a time when all surface values are nonzero, but the retardation still leads to successive equations. This also presents a new difficulty however. The derivation of the original Kirchhoff equation was valid only for continuous dependent variables. A discontinuous problem, e.g., scattering of a step wave, requires additional terms. Fortunately, these can be evaluated *in closed form*, e.g., a general discussion in Shaw (1968a) with examples in Friedman and Shaw, (1962) and Shaw and Friedman (1962).

An interesting sidelight developed in these last two references is the formulation of transient problems in terms of a total rather than a scattered field. The Kirchhoff retarded time equation as given above when applied to an exterior scattering problem must be written in terms of a dependent variable which decays appropriately at large distances from the scatterer to allow a closing surface 'at infinity' in Green's Theorem. This clearly is the scattered (total minus incident) field. The required boundary conditions could then be written on the scattered field. However, if such boundary conditions were originally homogeneous for the total field, they would now be inhomogeneous for the scattered field. This difficulty can be avoided by writing an integral equation on the incident field as well. This may

be developed in several ways. The most direct is to consider the boundary integral equation applied to the scattered field at a boundary point. The time harmonic case is used as an example; the approach is equally valid for the transient problem. The inhomogeneity, $f(\bar{r})$, is dropped for convenience.

$$(14)\quad (1/2)\,\phi_S(\bar{r}_B) = (1/4\pi)\int_B \left\{ \frac{\partial\phi_s(\bar{r}_0)}{\partial n_0}\cdot\frac{\exp[ikR]}{R} - \phi_s(\bar{r}_0)\frac{\partial}{\partial n_0}\left[\frac{\exp[ikR]}{R}\right]\right\} dA_0$$

Next, consider a formulation for the incident wave, $\phi_W(\bar{r})$. This is the field which would exist in the absence of the scattering body defined by the surface B. An exterior boundary integral equation on ϕ_W is not available since ϕ_W does not in general vanish at large distances from the scatterer and thus the integral over a closing surface 'at infinity' in Green's theorem may not vanish. However, an interior formulation is available; $\phi_W(\bar{r})$ may be expressed at some interior point to B, i.e., inside of the scattering surface, by considering the surface to be absent but by forming the integral equation over those values of ϕ_W and $\partial\phi_W/\partial n$ which exist at those points where the scattering surface was. This leads to a similar boundary integral equation as on ϕ_S as the interior point approaches the location of B, but with a change of sign for the normal direction. Keeping n_o the same as that in equation (9) leads to

$$(15)\quad (1/2)\,\phi_W(\bar{r}_B) = (-1/4\pi)\int_B \left\{ \frac{\partial\phi_W(\bar{r}_0)}{\partial n_0}\cdot\frac{\exp[ikR]}{R} - \phi_W(\bar{r}_0)\frac{\partial}{\partial n_0}\left[\frac{\exp[ikR]}{R}\right]\right\} dA_0$$

The total field, $\phi_T = \phi_W + \phi_S$, may be found by subtracting these equations, yielding

$$(16)\quad (1/2)\,\phi_T(\bar{r}_B) = \phi_W(\bar{r}_B) + (1/4\pi)\int_B \left\{ \frac{\partial \phi_T(\bar{r}_o)}{\partial n_o} \cdot \frac{\exp[ikR]}{R} - \phi_T(\bar{r}_o)\frac{\partial}{\partial n_o}\left[\frac{\exp[ikR]}{R}\right]\right\} dA_o$$

An alternative, and in some ways more useful, derivation may be described in the form of a 'saltus' problem. The exterior and interior wave problems for the same geometry are obviously closely related. If the domain in question is exterior to the boundary B, then ϵ is one for a field point exterior to B and zero for a field point interior to B and vice-versa for the interior problem, with a change in sign for the normal, n_o , which is always directed outward from the domain in question. The kernel functions, G and $\partial G/\partial n$, in the integral equation formulation may be interpreted as layers of monopole and dipole distributions of strength $\partial\phi/\partial n$ and ϕ respectively, which replace the boundary surface by an equivalent singular distribution in an infinite fluid domain including the interior of the body for the exterior case and vice-versa, i.e., a 'saltus' problem. These surface singularity distributions cause discontinuities in $\partial\phi/\partial n$ and ϕ of strengths $\partial\phi/\partial n$ and ϕ , i.e., they carry the value of ϕ and $\partial\phi/\partial n$ from their appropriate values on one side of the surface B to zero values on the other side. This property of distributions of monopoles and/or dipoles is well known and discussed in many references, e.g., Morse and Feshbach (1953). Now reconsider the equation on ϕ_S , at some field point $\bar{r}$ in the fluid writing ϕ_S and $\partial\phi_S/\partial n_o$ within the integral as the actual discontinuities in these values, i.e., $[\phi_S]$ and $[\partial\phi_S/\partial n]$ where $[\ ]$ is the value on the 'fluid' side of B minus the value on the 'empty' side of B (that side excluded from the Green's theorem integration). ϕ_S is replaced by $\phi_T - \phi_W$ and $[\phi_S]$, $[\partial\phi_S/\partial n_o]$ by $[\phi_T]$, $[\partial\phi_T/\partial n_o]$ since ϕ_W is continuous

everywhere in the fluid, even across B.

$$(17)\quad \phi_T(\bar{r}) - \phi_W(\bar{r}) = (1/4\pi) \iint_B \left\{ \left[\frac{\partial \phi_T(\bar{r}_o)}{\partial n_o} \right] \frac{\exp[ikR]}{R} - [\phi_T(\bar{r}_o)] \frac{\partial}{\partial n_o} \left[\frac{\exp[ikR]}{R} \right] \right\} dA_o$$

Now as the field point $\bar{r}$ approaches B, the integrals contribute a term $1/2[\phi_T(\bar{r})]$ leading again to Eq. (16), since ϕ_T inside B is zero identically, by this formulation.

The introduction of the concept of singularity distributions which replace the original problem by an equivalent saltus problem leads naturally in a discussion of the 'indirect' boundary integral formulations; the form above is called the direct approach since it deals with the actual dependent variables ϕ and $\partial\phi/\partial n$ 'directly'. The indirect approach is based on the observation that a distribution of monopoles or dipoles alone or in fact any combination of these will be sufficient to solve these problems; however, only the direct approach has the strengths of these distributions tied directly to the actual dependent variables of the problem. Excellent discussi[illegible] these alternative formulations are given by Chertock (1971) and by Kleinman and Roach (1974) in terms of the present formulations; many other classical discussions are also available, e.g., Lamb (1932)

Consider a monopole surface distribution of strength $\sigma(\bar{r}_o)$. The field at a point $\bar{r}$ is given by

$$(18)\quad \phi(\bar{r}) = (1/4\pi) \int_B \left\{ \sigma(\bar{r}_o) \cdot \frac{\exp[ikR]}{R} \right\} dA_o$$

where $\sigma(\bar{r}_o)$ must be $\partial\phi/\partial n_o - \partial\tilde{\phi}/\partial n_o$ where $\partial\phi/\partial n_o$ is the outward normal derivative of the exterior (interior) field under consideration

and $\partial\tilde{\phi}/\partial n_o$ is the normal derivative (in the same direction) of a complementary field; that which satisfied the same boundary values for ϕ for the interior (exterior) problem. For a problem with Dirichlet boundary conditions, this forms a 'boundary' integral equation on the monopole distribution strength $\sigma(\bar{r}_o)$ in terms of the prescribed $\phi(\bar{r}_o)$. There is no singular contribution as $\bar{r}$ approaches the surface B. Neumann boundary conditions could be treated by differentiating the equation on $\phi(\bar{r})$ in an n direction before approaching the surface; thus leads to an alternate form

$$(19) \quad \partial\phi(\bar{r}_o)/\partial n = \oint_B \left\{ \sigma(\bar{r}_o) \frac{\partial}{\partial n}\left[\frac{\exp[ikR]}{R} \right] \right\} dA_o - (1/2)\,\sigma(\bar{r}_o)$$

Similar results could be obtained for the dipole layer

$$(20) \quad \phi(\bar{r}) = \int_B \left\{ \mu(\bar{r}_o) \frac{\partial}{\partial n_o}\left[\frac{\exp[ikR]}{R} \right] \right\} dA_o$$

or any combination of monopole and dipole distributions. These alternate formulations have some advantages, as is most apparent in the solution of exterior time harmonic scattering at wave numbers near to the eigenvalues of the corresponding interior problem. The Fredholm alternative indicates no unique solution to the integral equation at these values, even though the exterior problem does have a unique and well behaved solution for all wavenumbers. This is due to the close coupling between boundary integral equation formulations of exterior and their complementary interior problems. This coupling has presented substantial difficulty in the application of BIE methods

to time harmonic exterior scattering problems as described for example by Chertock (1971), Kleinman and Roach (1974) which describe monopole and/or dipole combinations to circumvent this problem and has led to several 'modified' procedures, e.g., those of Schenck (1968), which uses additional equations for field points within the scatterer, Ursell (1973) which uses a modified wave source solution which satisfies a dissipative boundary condition on some surface interior to the scatterer. Of course, the BIE method is quite suitable for the determination of these eigenvalues for the interior problem, e.g., Tai and Shaw (1974), DeMey (1976), Hutchinson (1978); this is discussed further in a later section.

Finally, an alternative formulation to that usual boundary integral equation needs to be given. If $\partial\phi/\partial n$ is specified on B for equations (6) or (16), for example, the resulting integral equation is one of the second kind and numerical procedures may be used without too much difficulty. If however, ϕ is specified, i.e., a Dirichlet problem, the result is an integral equation of the first kind on the dependent variable, $\partial\phi/\partial n$. While this may be solved numerically, e.g., Shaw and English (1972) for transient scattering by a pressure release sphere, it may be useful to convert the original form into an equation of the second kind before numerical approximation. This is done by taking the field point off of the surface B, differentiating with respect to the normal direction n and then allowing the field point, at which $\partial\phi/\partial n$ is now defined, to approach B;

$$(21)\quad (1/2)\,\frac{\partial\phi(\bar{r}_B)}{\partial n} = \frac{1}{4\pi}\int_B \left\{ \frac{\partial\phi(\bar{r}_0)}{\partial n_0}\cdot\frac{\partial}{\partial n}\left[\frac{\exp[ikR]}{R}\right] - \phi(\bar{r}_0)\,\frac{\partial^2}{\partial n\,\partial n_0}\left[\frac{\exp[ikR]}{R}\right]\right\} dA_0$$

and an analogous equation for the transient case where such an approach was used by Shaw and Friedman (1962). Further discussion on the difficulty of numerical solution of Fredholm integral equations of the first kind is given for example in Delves and Walsh (1974).

NUMERICAL SOLUTION

Due to the large number of examples available in literature, as given in part in the references and bibliography, and to the several review articles available, it does not seem appropriate to give details of either numerical procedures or actual programs. In the sense that this is still a developing field, any of the numerical techniques and/or computer programs are subject to significant improvement. What does appear appropriate here is a description of the forms of numerical solution techniques and those pitfalls which have been encountered and, where this has been found possible, circumvented.

Consider first the class of interior time-harmonic problems for a finite region bounded by B where some suitable boundary conditions are given. Here the homogeneous and inhomogeneous problems must be distinguished. The former has as its goal the determination of the eigenvalues and eigenmodes for that region with those particular boundary conditions, e.g., Tai and Shaw (1974), DeMey (1976), Gregoire, Nedelec and Planchard (1975), Hutchinson (1978). As an example, consider the two dimensional resonance of an acoustic domain with a rigid boundary, i.e., a Neumann boundary condition. The governing homogeneous boundary integral equation is

$$(22)\quad (1/2)\phi(\bar{r}_B) = (i/4)\int_B \phi(\bar{r}_0)\frac{\partial}{\partial n_0}\left[H_0^{(1)}(kR)\right]dA_0$$

In the simplest approach, ϕ is approximated by a piecewise constant over N segments of B, preferably equal in size, and the field point, $\bar{r}_B$, is placed at the midpoint of each segment with $\phi(\bar{r}_B)$ for $\bar{r}_B = \bar{r}_K$ defined as the constant for that segment, ϕ_K. Then

$$\frac{1}{2}\phi_K = \sum_{L=1}^{N} \phi_L \cdot \alpha_{KL}, \quad K = 1, 2, \cdots N \tag{23}$$

with

$$\alpha_{KL} = (i/4) \int_{B_L} \frac{\partial}{\partial n_0}\left[H_0^{(1)}(k|\bar{r}_0 - \bar{r}_K|)\right] ds_0 \tag{24}$$

The principle value occurs only for that coefficient where the interval L includes the field point K, i.e., L equals K. All other coefficients are calculated as ordinary integrals. The integral for α_{KK} excludes the point $\bar{r}_o$ equal to $\bar{r}_K$; an examination of the form of the integrand shows it to be $-kH_1^{(1)}(kR)\partial R/\partial n_o$. As a first approximation, consider α_{KK} to be evaluated by any numerical scheme, e.g., a trapezoidal rule, in which the (curved) segment B_L is replaced by a (large) number of straight subsegments, e.g., the intervals used in the trapezoidal rule. That straight segment containing the field point, i.e., R equal zero, also has $\partial R/\partial n_o$ equal to zero everywhere. Since the point R equal zero is excluded from the integration (this is a principle value integral), the contribution of this subelement to the coefficient α_{KK} is zero. Of course, for a simple curved geometry, the integral might be evaluated in closed form as a principle value integral directly, but this is unlikely for an arbitrary shape. It is useful from a numerical view to take the subsegments in the integration

over B_L such that the actual field point falls at the midpoint of one of these subsegments. Details for the calculation of these 'influence coefficients' (which are also used in forced problems) can be found in almost any of the references, e.g., Banaugh and Goldsmith (1963), for a somewhat more accurate evaluation.

The resulting system of linear algebraic equations must have a zero determinant for the coefficient matrix, i.e.,

$$\left| (1/2)\, \delta_{KL} - \alpha_{KL} \right| = 0$$

where α_{KL} are functions of the parameter k, to have a nontrivial solution. Those values of k satisfying this equation are then the eigenvalues of the original problem.

Some care must be taken with the form of the Green's function used. Some authors, deMey (1977) and Hutchinson (1978), have suggested that a simplification could be achieved by using $Y_o(kR)$ in place of $H_o^{(1)}(kR)$ as the fundamental solution. This possesses the appropriate singular behavior and satisfies the Helmholtz equation and furthermore leads to real equations (real α_{KL}) on the (real) eigenvalues, k. Unfortunately, this apporach is incorrect since the $Y_o(kR)$ fundamental solution contains both outgoing <u>and</u> incoming waves whereas the $H_o^{(1)}(kR)$ alone possesses the appropriate behavior of outgoing waves at large R. This can be readily seen in a simple example, a circular boundary, to which the solutions are well known. In this particular case, described in detail in the report version of Tai and Shaw (1974), the boundary integral equation becomes

$$(25) \quad (1/2)\, \phi(\theta) = (ika/4) \int_0^{2\pi} \phi(\theta + 2\beta)\, H_1^{(1)}(2ka \sin\beta) \sin\beta \, d\beta$$

where $\beta = (\theta-\theta_0)/2$. An expansion of $\phi(\theta)$ as a Fourier series and a corresponding modal separation gives

$$\phi(\theta) = \sum_{m=0}^{\infty} \left[A_m \sin(m\theta) + B_m \cos(m\theta) \right]$$

with

$$A_m = \left(\frac{ika}{2}\right) \int_0^{2\pi} A_m \cos(2m\beta)\, H_1^{(1)}(2ka\sin\beta) \sin\beta \, d\beta$$

and

$$B_m = \left(\frac{ika}{2}\right) \int_0^{2\pi} B_m \cos(2m\beta)\, H_1^{(1)}(2ka\sin\beta) \sin\beta \, d\beta$$

These equations are identical; clearly either A_m or B_m could vanish, but not both for a nontrivial solution. Thus, in real and imaginary parts,

$$(2.6)(a) \quad \underline{1} = -ka \int_0^{2\pi} \cos(2m\beta)\, Y_1(2ka\sin\beta) \sin\beta \, d\beta = 1 + \pi ka\, Y_m(ka) J_m'(ka)$$

$$(b) \quad 0 = ka \int_0^{2\pi} \cos(2m\beta)\, J_1(2ka\sin\beta) \sin\beta \, d\beta = \pi ka\, J_m(ka) J_m'(ka)$$

The only common root is $J_m'(ka)$ equal to zero and these are of course the correct colutions (apart from ka itself zero). Note however that the use of the $Y_0(kR)$ fundamental solution would have led to only the first equation which has an additional set of spurious roots, $Y_m(ka)$ equal to zero. Thus it is necessary to keep the original fundamental solution and pay the price of the complex determinant calculation in order to obtain the appropriate eigenvalues.

The corresponding eigenmodes can be calculated from the original set of algebraic equations by choosing a reference value for one of the

values ϕ_K , e.g., ϕ_1 equal 1. As long as this is not a node, the resulting set of N equations on N-1 unknowns may be solved for the eigenmode on the boundary; interior values may be obtained from the original Helmholtz or Weber integral equation with the field point interior to B, i.e.,

$$\phi(\bar{r}) = (i/4) \int_B \phi(\bar{r}_0) \frac{\partial}{\partial n_0} \left[H_0^{(1)}(kR) \right] ds_0$$

where this is a simple quadrature since $\phi(\bar{r}_0)$ on B is now known from the previous calculation.

Finally some comment should be made on the use of boundary integral equation methods for higher modes. To accurately represent a rapidly varying function, many boundary segments would be required leading to a large system of algebraic equations and thus a high order determinant. This numerical approach then appears unwieldly; however it is for these higher modes (large values of ka) that the asymptotic solutions to the boundary integral equation may play a significant role.

If Dirichlet boundary conditions are given, the resulting boundary integral equation is one of the first kind. These are inherently less stable for numerical solution than those of the second kind, e.g., Delves and Walsh (1974), and should probably be converted over to equations of the second kind, as described above in the section on formulation, before proceeding with numerical solution.

The interior inhomogeneous time harmonic problem has not been as widely studied as the corresponding exterior problem, which as great application in terms of wave scattering and radiation in our infinite domain. Indeed, this is probably the most widely studied case for the boundary integral equation method as applied to wave problems.

The major difficulty with this group of problems is not the actual numerical solution procedure but rather the inherent difficulty in obtaining solutions to the exterior boundary integral equations at wave numbers which match eigenvalues for a corresponding (but altered boundary condition) interior problem, e.g.,

Exterior Dirichlet → Interior Neumann eigenvalues

Exterior Neumann → Interior Dirichlet eigenvalues

This is not a physical difficulty inherent in the exterior problem since there are no 'exterior' eigenvalues for these acoustic problems, although such questions do arise for long period water waves over variable topography, e.g., Meyer (1971). Rather this is a difficulty inherent in the boundary integral equation approach, as discussed by a number of authors, with a corresponding number of methods of circumventing the problem, as described in the previous section. To illustrate this point, consider the example of a two dimensional circle of radius a radiating into an infinite acoustic domain with a prescribed time harmonic surface velocity, $v_n = -\frac{\partial\phi}{\partial n} = v_o$. For simplicity take the axisymmetric case, i.e., v_o is constant. Then equation (6) becomes

$$(27) \quad \epsilon\phi(\bar{r}) = (i/4)\int_0^{2\pi}\left\{-v_o \cdot H_o^{(1)}(kR) - \phi(\theta_o)\frac{\partial}{\partial n_o}\left[H_o^{(1)}(kR)\right]\right\} a\, d\theta_o$$

where ϕ is a constant ϕ_o on the surface, r = a, i.e.,

$$\epsilon\phi(\bar{r}) = -\frac{i a v_o}{4}\int_0^{2\pi} H_o^{(1)}(kR)\, d\theta_o - \frac{i a \phi_o}{4}\int_0^{2\pi}\frac{\partial}{\partial n_o}\left[H_o^{(1)}(kR)\right] d\theta_o$$

The field point is __not__ placed on the surface since the expansion of the fundamental solution, as given in Morse and Feshbach, sec. 7.2 (1953), requires a different value for $|\bar{r}|$ and $|\bar{r}_o|$.

$$R = \left[r^2 + r_0^2 - 2 r r_0 \cos(\theta - \theta_0) \right]^{1/2}$$

$$\partial R / \partial n_0 = - \partial R / \partial r_0 = - \left(r_0 - r \cos(\theta - \theta_0) \right) / R$$

$$H_0^{(1)}(kR) = \sum_{m=0}^{\infty} \epsilon_m \cos(m(\theta - \theta_0)) \begin{cases} J_m(kr)\, H_m(kr_0) \ ; & r < r_0 \\ J_m(kr_0)\, H_m(kr) \ ; & r > r_0 \end{cases}$$

where $\varepsilon_m = 1$ if $m = 0$ and 2 if $m \neq 0$.

$$\frac{\partial H_0^{(1)}(kR)}{\partial r_0} = k \sum_{m=0}^{\infty} \epsilon_m \cos(m(\theta - \theta_0)) \begin{cases} J_m(kr)\, H_m'(kr_0) \ ; & r < r_0 \\ J_m'(kr_0)\, H_m(kr) \ ; & r > r_0 \end{cases}$$

For simplicity, take $\theta = 0$ and $r = a^-$; since $r_o = a$, this places r infinitesimally less than r_o and also places the field point exterior to the infinite fluid, i.e., interior to the circular boundary. Then $\varepsilon = 0$ and

$$0 = - \frac{i a N_0}{4} J_0(ka^-)\, H_0(ka) - \frac{i a \phi_0}{4} k\, J_0(ka^-)\, H_0'(ka)$$

or

$$\phi_0 = - \frac{N_0}{k} \cdot \frac{H_0(ka)}{H_0'(ka)} \qquad (28)$$

which agrees with the well known separation of variables solution <u>if</u> $J_o(ka^-)$ is not zero, i.e., if k is not an eigenvalue to the corresponding Dirichlet problem.

Apart from this very real but treatable difficulty, the two and three dimensional time harmonic exterior problems are no more difficult than the interior counterparts; this is in strong contrast to the alternative numerical methods of finite differences and finite elements.

Transient problems form the next class for discussion. As mentioned above, the two dimensional transient fundamental solution contains a time integration and it has been found convenient to treat two dimensional problems in terms of an extension into a three dimensional form

with infinite cylinders of constant cross section replacing the original two dimensional body, e.g., Friedman and Shaw (1962). There have been a number of two and three dimensional problems considered; these have been exclusively exterior problems (see the references and bibliography). The main advantage of time dependent boundary integral equation solutions is that they lead to uncoupled or weakly coupled systems of algebraic equations when Φ is approximated by some trial function, e.g., a piece wise constant, over the boundary elements. Furthermore, solutions for discontinuous forcing functions, e.g., shock wave or step pressure loading, can be treated with the contribution from the discontinuity evaluated exactly, e.g., Friedman and Shaw (1962), Shaw (1968c). While integral equation formulations of the second kind are preferable, e.g., Shaw and Friedman (1962), some solutions have been carried out using integral equation formulations of the first kind, e.g., Shaw and English (1972).

As a simple illustration, consider the two dimensional scattering of a step plane pressure pulse by an infinite right angle wedge with a rigid boundary condition; this forms the 'front' portion of the problem described by Friedman and Shaw (1962). The problem is treated as three dimensional with z the coordinate axis in the third direction. The two surfaces of the wedge are taken as the xz and yz planes and the governing equation for a field point at (x,o,o) is then

$$(29)\quad (1/2)\,\phi(x,0,0,t) = \phi_w(x,0,0,t) + \frac{1}{4\pi}\int_B \left\{ \frac{1}{cR}\frac{\partial\phi(\bar{r}_0,t_0)}{\partial t_0} + \frac{\phi(\bar{r}_0,t_0)}{R^2} \right\} \frac{\partial R}{\partial n_0}\, dA_0$$

Here S represents both surfaces but since $\partial R/\partial n_o$ vanishes along

the $x_o z_o$ surface (the singular contribution from the point R equal zero has already accounted for, i.e., the principle value excludes this point) only integrations on the $y_o z_o$ plane remain. Taking a symmetric incident wave for simplicity, the solution on the $y_o z_o$ surface is equal to that on the $x_o z_o$ surface and the equation becomes

$$(30)\quad \frac{1}{2}\phi(x,0,0,t) = \phi_W(x,0,0,t) + \frac{1}{4\pi}\int_0^{y_{OU}}\int_{z_{OL}}^{z_{OU}}\left\{\frac{1}{cR}\frac{\partial\phi(\bar{r}_0,t_0)}{\partial t_0} + \frac{\phi(\bar{r}_0,t_0)}{R^2}\right\}\frac{\partial R}{\partial n_0}\,dz_0\,dy_0$$

The limits of integration are determined by the condition

$$(31)\quad \left[ct - (\sqrt{2}/2)y_0\right]^2 = x^2 + y_0^2 + z_0^2$$

which yields

$$(32)\ a)\quad y_{OU} = ct/(1+\sqrt{2}/2)$$

$$b)\quad z_{OU} = -z_{OL} = \left[(ct - (\sqrt{2}/2)y_0)^2 - y_0^2\right]^{1/2}$$

$$= 0 \text{ at } y_{OU}$$

Thus, equation ( ) has time dependent limits of integration. If an incident pressure field, $p_W = \rho_o\,\partial\phi/\partial t$, is prescribed as a unit step function, $H[ct-(x+y)\sqrt{2}/2]$, this equation on the velocity potential ϕ may be differentiated, using Leibnitz' rule to interchange the spatial integration with the temporal differentiation

$$(33)\quad \frac{1}{2}\phi(x,0,0,t) = \phi_W(x,0,0,t) + \frac{\rho_0}{4\pi}\int_0^{y_{OU}} 2\,\frac{\partial z_{OU}}{\partial t}\left[\frac{\phi(\bar{r}_0,t_0)}{R^2} + \frac{1}{cR}\frac{\partial\phi(\bar{r}_0,t_0)}{\partial t_0}\right]_{z_0=z_{OU}}\frac{\partial R}{\partial n_0}\,dy_0$$

$$+ \frac{1}{4\pi}\int_0^{y_{OU}} 2\int_0^{z_{OU}}\left[\frac{\phi(0,y_0,z_0,t_0)}{R^2} + \frac{1}{cR}\frac{\partial\phi(0,y_0,0,t_0)}{\partial t_0}\right]dz_0\,dy_0$$

The first integral is evaluated at z_{ou} which is immediately behind the wave front; ϕ and $\partial\phi/\partial t$ are known here as the direct reflected solution; ϕ is zero and $\partial\phi/\partial t$ is $2/\rho_o$. Thus this term may be integrated as a simple quadrature, yielding a function $A(x,t)$.

The numerical solution of this boundary integral equation may be accomplished by taking p to have a constant value, $p_{K,L}$, for $(K-1)\Delta Y<Y_o<K\Delta Y$ and for $(L-1)\Delta t<t<L\Delta t$. The time derivative may be replaced by a backwards time difference,

$$(34)\quad \frac{\partial p(0,y_o,z_o,t_o)}{\partial t_o} = \frac{p(0,y_{oK},z_o,t_{oL})-p(0,y_{oK},z_o,t_{oL}-\Delta t)}{\Delta t} = \frac{p_{K,L}-p_{K,L-1}}{\Delta t}$$

The location of the field point, x_j, enters through the term R in the retarded time t_o. Define

$$(35)\ a)\quad \beta_{KLJ} = (1/2\pi)\int_{B_{KL}} (1/R^2)\,\partial R/\partial n_o \, dy_o dz_o$$

$$b)\quad \gamma_{KLJ} = (1/2\pi)\int_{B_{KL}} (1/cR)\,\partial R/\partial n_o \, dy_o dz_o$$

where B_{KL} is the surface element defined by $(K-1)\Delta Y<Y_o<K\Delta Y$ and $(L-1)\Delta t<c<L\Delta t$, such that $(M-L)\Delta t<t_o<(M-L+1)\Delta t$ there. It is simpler to use the condition $ct = R$ in place of equation (31); this would have source points contributing to the integral before an appropriate time interval has passed allowing these points to be stimulated by the incident wave and then send a signal to the field point. As long as the strength of these contributions is zero, this integral in theory does not change. In practice, there is a small effect due to the finite discretization of the integral but this effect is negligible. It would not have been appropriate however to make this modification before the wave front discontinuity contribution, A(x,t), was calculated.

The boundary integral equation is then replaced by

$$(36)\quad (1/2)\, p_{J,M} = p_W(x_J, t_M) + A(x_J, t_M) + \sum_{K=1}^{N} \sum_{L=1}^{M} \left\{ \beta_{KLJ}\, p_{K,M-L+1} + \gamma_{KLJ} \left[\frac{p_{K,M-L+1} - p_{K,M-L}}{\Delta t} \right] \right\}$$

where X_j is chosen at a 'midpoint', $(J-1/2)\Delta X$ and t_M at an 'endpoint', $(M-1)\Delta t$ *of the space and time step respectively, with* $\Delta Y = \Delta X = \sqrt{2}\, c\Delta t$, to allow the incident wave to sweep over one space step in one time step.

The solution of this set of linear algebraic equations is considerably simplified when they are recognized to be successive rather than simultaneous, i.e., the value p_{JM} does not appear on the right hand side of equation (36) since β_{KLJ} and γ_{KLJ} are both zero for negative L and p, $\dot{p}$ are zero for zero and negative time, i.e., $M \leq L - 1$. If t_M were chosen to be $M\Delta t$, i.e., the value of p at the end rather than the beginning of the M^{th} time step, there would be a weak coupling in that β_{J1J} and γ_{J1J} would both multiply values of p_{JM} on the right hand side of equation (36). These can be easily moved over to combine with the (1/2) p_{JM} term; the remainder of the summation does not involve values of p at the present time step. Choosing $c\Delta t < \Delta X$ or ΔY should insure stability for an explicit scheme were this a finite difference scheme; the appropriate stability criteria here do not appear to have been studied. Clearly a large time step (covering several space steps) would lead to a strongly coupled (implicit) scheme while a small time step (covering a fraction of a space step) would lead to an uncoupled system. The time step and space step used in much of this work is matched to the incident wave , a step size ratio of $c\Delta t/\Delta X$ equal to or less than one should be used to remain within the classical Courant, Lewy, Fredricks stability criteria. Numerical results are given in Friedman and Shaw (1962) and agree very well with

an analytical solution. Before leaving this problem however, it is of interest to note that the vertex (o,o) lies on 'both' surfaces and has $\partial R/\partial n_o$ equal to zero on both surfaces, thereby causing the surface integral in equation (29) to vanish except for the singular contribution at $R = 0$ which produces ε equal to 3/4 instead of 1/2 (the region excluded from the integration in Greens Theorem is 3/4 of a circle rather than 1/2 of a circle for this boundary point). The analytical solution at this vertex is then seen to be $\phi(o,o,t) = (4/3)\phi_w(o,o,t)$ for any form of incident wave. Furthermore, boundary points for $\sqrt{2}\,ct > X > ct$ have a solution $\phi(X,o,t) = 2\phi_w(X,o,t)$ since the double integral is zero for these values even though these points have been reached by the incident wave; this is simply the reflection solution before corner diffraction effects reach the field point.

The simple numerical cases described above give the essence of the procedures used. Improvements clearly can be made in the form of the trial functions and boundary element shapes used, e.g., Cruse (1974) and, using finite element methods to solve boundary integral equations, Jeng and Wexler (1977).

Alternatively, Galerkin or other moment methods could be employed, e.g., Sharma (1967). These improvements, though an important part of the progress of this field, are not in themselves necessary for a basic understanding of the 'mechanics' of this numerical approach.

APPROXIMATE SOLUTIONS

Although the emphasis of this chapter, and indeed of this text is on numerical solution methods, this section appears appropriate in

that it does represent a solution technique valid for certain types of problems. These problems generally fall in that class which lends itself to some form of expansion in terms of a physical parameter which is small. Typical of such solutions are the studies of time harmonic wave radiation by shells forms of slightly nonconcentric cylindrical surfaces, submerged in an infinite acoustic fluid, Shaw and Tai (1974) and Shaw (1978a). Here an expansion of the problem may be made in powers of the nonconcentricity distance; when this distance is zero, the outer and inner shell surfaces are concentric and the problem lends itself to separation of variables. This technique can be readily applied to other geometries, e.g., spherical, ellipsoidal, etc., in which the Helmholtz equation separates. The details of this procedure are left to the references; they basically involve an expansion of the fundamental solution into the modes of the separable solution and simultaneously a further expansion into powers of the separation distance. Consideration of terms of like order in this (small) distance leads to a successive system of equations which are of the same form for each order with only a modified 'forcing' term.

Another area of approximate solutions, which overlaps into the use of boundary integral equation methods for analytical purposes, is their use in deriving elastic plate or elastic shell theory from three dimensional elasticity. If one considers that the ultimate goal of plate (or shell) theory is to provide a valid two dimensional representation of an inherently three dimensional problem, the role of boundary integral equation methods is apparent. They provide this service <u>without</u> approximation. Since there are in general two distinct surfaces (upper and lower or inner and outer) with possibly other edge surfaces over which the boundary integral is formed, it is clear that the plate (shell)

thickness will enter into the kernels (Green's functions) through their dependence on the distance from field point to integration (source) point. An expansion of these kernels in terms of this thickness allows a heirarchy of plate (shell) theories to be developed from an essentially rigorous foundation in which this parameter occurs naturally. Since this application is somewhat removed from the main stream of this chapter, details are left to a reference, Shaw (1978b). It would be inappropriate however to leave this subject without indicating that St. Venants principle follows very naturally from a boundary integral equation formulation, e.g., Shaw (1978c); indeed, a remarkable proof of this principle for the elastic half space was given by Sternberg (1954) using a form of Betti's theorem which in turn is a basis for an boundary integral equation formulation of elastostatics, Cruse and Rizzo (1968, Cruse (1968).

Equally far removed from this mainstream are the high frequency approximations of optics and acoustics, i.e., geometrical optics (acoustics). Where these techniques are employed in the treatment of wave propagation through inhomogeneous media, boundary integral equations as yet offer little competition in view of the lack of appropriate fundamental solutions.

However, boundary integral equation methods may play a significant role in the study of high frequency scattering by complex structures -- especially in terms of an expansion of the integral equation kernels in terms of the (small) parameter, the wavenumber. This approach formed the basis of much classical analysis, e.g., Fresnel-Kirchhoff diffraction theory (which contained other assumptions as well).

CONCLUSIONS AND DIRECTIONS

Probably the most useful discussion of the application of boundary integral equation methods to wave propagation problems is that of the directions in which these seem to be heading. This is best stated in terms of where these methods have been used already and conclusions as to their present status. As with most of this chapter, these are subjective to some extent but should provide a reasonable overview.

In acoustics, two and three dimensional time harmonic exterior radiation and scattering problems seem well in hand e.g., reviews by Shaw (1970), Chertock (1971), Kleinman and Roach (1974). Primary advances here would seem to be in the areas of improved trial functions, since most examples to date use a simple constant form, and surface segmenting and in the solution of the resulting system of algebraic equations -- problems analogous somewhat to those currently found in the area of finite elements. It seems quite feasible that an all purpose program can be developed for arbitrarily shaped surfaces. Some care should be taken however in realizing that the strength of numerical solution by boundary element discretization lies in the area where classical high frequency and low frequency procedures are inadequate, i.e., for wavelengths comparable to the length scale of the obstacle. Finally more work is required on eliminating those spurious results which arise from the complementary interior problem eigenvalues, e.g., Urcell (1975), Schenck (1968). In fact the use of boundary integral equation methods to determine interior problem eigenvalues, e.g., Tai and Shaw (1974), DeMey (1976), seems another area worth further examination.

Transient acoustic problems seem to also be off to a good start with direct Kirchhoff retarded time integral equation solution methods

suitable for early time, e.g., Friedman and Shaw (1962), Shaw and Friedman (1962), Mitzner (1967), and Fourier transform and corresponding Helmholtz integral equation solutions for longer time, e.g., Shippy (1975). More complex geometries and eventually a general numerical program for arbitrarily shaped obstacles seem to be the next steps.

The analogous elastic problems have a further dilemma. Should their formulation be in terms of displacement potentials, which have simple governing equations but complicated boundary relationships, e.g., Shaw (1968), or directly in terms of tractions and displacements which have simpler boundary representations but more complicated governing equations (and thus kernel functions), e.g., Cruse and Rizzo, (1968), Cruse (1968).

Coupled acoustic fluid-elastic structure problems seem a natural application of boundary integral equation methods, due to their common boundary, especially when the acoustic fluid is infinite in extent. This is true whether the elastic structure is represented by another boundary integral equation, e.g., Wilkinson and Liu (1970), Shaw (1973a), Shaw (1973b) or by some other rperesentation such as finite elements.

Water wave problems form still another class of problems only briefly touched on here. For long wave theory, the governing equation is analogous to a two dimensional acoustic wave equation. However, the phase velocity is dependent on the water depth which clearly varies in such problems and thus the analogy is to acoustic propagation in

an inhomogeneous medium. This is further complicated by the treatment of most of these as exterior problems, i.e., involving an infinite domain. This has lead to the introduction of coupled boundary integral equation -- other numerical methods, e.g., Shaw (1974c), Shaw (1975a), to finite differences, Shaw and Falby (1977), Shaw (1978) to finite elements, etc. for problems where the infinite domain is represented by some constant depth (constant phase velocity) at some distance from the local inhomogeneity. Indeed, this coupling is an area of general interest, e.g., Zienkiewicz, Kelly and Bettess (1977). For waves over submerged obstacles, a boundary integral equation approach has been found useful based on a somewhat complicated fundamental solution and has been used by Garrison and Seetharama (1971), Garrison and Chow (1972), etc. with success. One of the primary difficulties here is that the governing equation is simple -- indeed it is the Laplace equation -- but the free surface boundary conditions are difficult. In fact, they may be taken as nonlinear and still yield to a boundary integral equation approach since the governing differential equation is linear, e.g., Shaw (1975).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An introduction is given to the application of boundary integral equation methods to the areas of wave propagation, radiation and scattering. Such applications were among the first for this method which made use of the power of modern digital computers; however, there is a long history of integral methods in this field. The emphasis here is on fundamental concepts coupled with illustrations of the application of this method. The breadth of the applications of this approach precludes any attempt at overall completeness ...

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20. Abstract (continued)

and many details are left to an extensive, but undoubtedly incomplete, list of references. There is however sufficient material to formulate the solution of most problems, with some of the inevitable pitfalls indicated.